Notes on Smoothing Theory

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References we have are: For Top to Smooth we have Kirby-Sieberman and Aru-Powels notes on the first three essays [ACE$^+$], [KS16]. Milnor in [Mil64] sets up the basics. The early results are comparing PL and Diff in [Hir63], [HM] and [LR65]. There is also the paper [BL74] for an early proof of Morlet. Modern reinterpretations of these results are in [ST26] and [KK24]. There is also the notes from Luries course on Geometric Topology. There is also the book [Rud14].

The classification of smooth structures comes in terms of the space of lifts of the 'topological' tangent bundle. Statements about bundles are often usefully phrased in terms of their classifying spaces, and moreover calculations of the classifying spaces allow one to actually *apply* the classification. So the classification can be stated without classifying spaces but its less useful.

1 Background

1.1 Micro-bundles

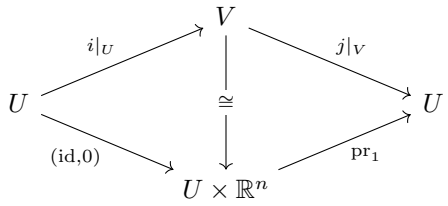
To get started we will have to have in mind definitions of smooth, peice wise linear and topological manifolds. A topological manifold is a (paracompact Hausdorff) topological space with a *property*, namely that it is locally Euclidean. PL and smooth manifolds are topological manifolds with a *structure*, namely a PL or smooth atlas. The existence of a topological atlas would be vacuous since the inverse of a homeomorphisms is always a homeomorphism (whereas smooth and PL maps can fail on overlaps). The first object of smoothing theory will be micro-bundles, meant to mirror the structures of a zero section (i) and a projection (j). To be category agnostic we let $\mathrm{CAT} \in \{\mathrm{Diff}, \mathrm{Top}, \mathrm{PL}\}$ and maps will have to be interpreted appropriately.

For completeness recall that a we call an injective map $K \to \mathbb{R}^n$, where $K \subseteq \mathbb{R}^n$ is compact, PL if K can be written as a finite union of simplicies and the map is affine on each one. A map $\mathbb{R}^n \to \mathbb{R}^n$ is affine if its of the form $f(x) = Ax + b$, where A will be a matrix.

Definition ([ACE$^+$] Def 8.2)**.** *A* CAT *microbundle* $\mathfrak{X}$ *of fiber dimension* n *consists of a* CAT *base space* B *and total space* E *and a pair of* CAT *maps*

$$B \xrightarrow{i} E \xrightarrow{j} B$$

such that $j \circ i = \mathrm{id}_B$ *and that satisfy: for all* $b \in B$ *there exists* $b \in U, i(b) \in V$ *opens and a* CAT *isomorphism* $V \xrightarrow{\cong} U \times \mathbb{R}^n$*, such that* $i(U) \subseteq V, j(V) \subseteq U$ *and the diagram commutes*



Example (Trivial Micro-budle)**.** *Over any space* B *there is a trival microbundle of dimension* n *which may be denoted* $\mathfrak{e}^n_B$*, given by*

$$B \xrightarrow{(\mathrm{id},0)} B \times \mathbb{R}^n \xrightarrow{\mathrm{pr}_1} B.$$

To see that there exist opens satisfying the condition we need only take the entirty of B *and* $B \times \mathbb{R}^n$*.*

Example (Underlying Micro-budle)**.** *If* $p : E \to B$ *is a* CAT $\mathbb{R}^n$ *fiber bundle then the zero section and projection can be taken as the* i, j *respecively and moreoever the opens for the condition can be taken to be the charts of the local triviality condition for the fiber bundle.*

Example (Tangent Micro-budle)**.** *For a* CAT *manifold* M *there is a microbundle given by the diagonal*

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\mathrm{pr}_1} M$$

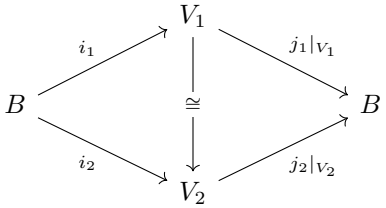
To see that the existence condition is satisfied let $x \in M$ *then* $x \in U$ *and* $f : U \xrightarrow{\cong} \mathbb{R}^n$ *be a* CAT *chart around* x*. Then*

$$h : U \times U \to U \times \mathbb{R}^n$$

$$(u, v) \mapsto (u, f(v) - f(u))$$

is a CAT *isomorphism. Notice that we have to take the difference to ensure that the left triangle commutes,* $h\Delta(x) = (x, 0)$*. This micro-bundle clearly uses the fact that this is a manifold (charts), we call it the 'tangent micro-bundle' and denote it* $\mathfrak{t}_M$*. Even though the total space seemingly has a lot of topology, the fiber being all of* M*, as is clear from checking the condition the idea is to consider only a neighbourhood of the zero section.*

Definition ([ACE$^+$] Def 8.7)**.** *Two micro-bundles* $\mathfrak{X}_1, \mathfrak{X}_2$ *over the same base* B *are equivalent if there exists neighbourhoods* $i_n(B) \subseteq V_n \subseteq E_n$ $n = 1, 2$ *and* CAT *isomorphisms* $V_1 \to V_2$ *such that the following diagram commutes*



Note that there are non-isomorphic smooth vector bundles whose underlying topological micro-bundles are isomorphic, thus this is a weaker sort of structure. Notice that this definition of equivalence does not invoke the total space, but only a neighbourhood of the zero section. The following theorem somewhat justifies the use of the term tangent microbundle,

Theorem ([ACE$^+$] Thm 8.9)**.** *If M is a smooth manifold then there is an isomorphism of microbundles $TM \cong \mathfrak{t}_M$, where $\mathfrak{t}_M$ is the* CAT *tangent micro-bundle.*

Proof. (Sketch) If CAT = Top then recall the exponential map provides a map $TM \to M \times M$ that is locally an homeomorphism around the zero section, modelled locally on $(p, v) \mapsto (p, \exp(p, v))$. This provides a local homeomorphism around each point in M that can be upgraded to a homeomorphism globally for $M \subseteq TM$ by patching these neighbourhoods.

If CAT = Diff then applying the tubular neighbourhood theorem to the zero section. □

We have seen that every $\mathbb{R}^n$ fiber bundle has an underlying micro-bundle. Kisters theorem is a converse for manifolds, Kister proved the topological case, the smooth and PL were done seperately but are much easier.

Theorem ([ACE$^+$] Thm 8.10 for Top, [Kup] 27.2.6)**.** *Given a rank n* CAT *microbundle $\mathfrak{X}$, exhibited by $M \xrightarrow{i} E \xrightarrow{j} M$ with base space a* CAT *manifold, then there exists a neighbourhood of the zero section $i(M) \subseteq F$ making the projection $F \xrightarrow{j} M$ an $\mathbb{R}^n$ fiber bundle. The underlying microbundle of F is again isomorphic to $\mathfrak{X}$ and the fiber bundle is unique up to isomorphism.*

Proof. (Barely a sketch) The proof is quite involved. Locally microbundles are trivial, glue these peices together in the right way. To do so we need to make sure they are compatable on collars of the boundary of the patches, by an inductive argument on cells this is possible. Assume that you can make get the bundle on $m-1$ cells allows you to show it for m cells whose boundaries are patchable because they are $m-1$ cells. By [Kup, 27.0.2] there is a homtopy equivalence for CAT = Diff, Top

$$\mathrm{Aut}_{\mathrm{CAT}}(\mathbb{R}^n, 0) \hookrightarrow \mathrm{Emb}_{\mathrm{CAT}}(\mathbb{R}^n, 0)$$

when attempting to patch we need to argue that a certain map on the overlap is a homeomorphism, as in the definition of fiber bundle we require local trivialisations to be homeomorphisms. The induction hypothesis gives a trivialised bundle on the collar and we can shrink the given microbundle over the collar to embedd it in this trivialised bundle. *The proof* of this homotopy equivalence gives that we can isotope it to a homeomorphism (the proof does better than just homotopy equivalent). □

This can be interpreted as an equivalence for manifolds M,

$$\pi_0\Big[\mathbb{R}^n \text{ Fiber Bundles}/M\Big] \underset{F}{\overset{\text{underlying}}{\rightleftarrows}} \pi_0\Big[\text{micro-bundles}/M\Big].$$

There is a corresponding space level fact given in [AF15, Cor 2.4]. The moral of the story is then that micro-bundles are a superflous construction in the context of smoothing theory, at least in principle. On the other hand we need *a definition* of tangent bundles in the PL and Top cases and this is a framework with an elegant answer. One can see from the proof that the corresponding fiber bundle might be highly non-canonical or hard to construct. At this point it is possible to show that

Theorem ([ACE$^+$] 9.19)**.** *If M is a topological manifold then there exists a q such that $M \times \mathbb{R}^q$ has a smooth strucuture iff $\mathfrak{t}_M$ is stably isomorphic to the underlying microbundle of a vector bundle.*

Proof. (Sketch.)($\implies$) First the forward direction. Assume $M \times \mathbb{R}^q$ has a smooth structure. First note that because $\mathfrak{t}_{M\times N} \cong \mathfrak{t}_M \times \mathfrak{t}_N$ because we can take the neighbourhood in the definition to be the total spaces with the homeomorphism the obvious swapping map: $(M \times N) \times (M \times N) \xrightarrow{\sim} (M \times M) \times (N \times N)$. Then ($\mathcal{U}$ for underlying),

$$\mathcal{U}[T(M \times \mathbb{R}^q)] \cong \mathfrak{t}_{M\times\mathbb{R}^n} \cong \mathfrak{t}_M \times \mathfrak{t}_{\mathbb{R}^q} \cong \mathfrak{t}_M \times \mathfrak{e}^n_{\mathbb{R}^q}$$

The first isomorphism was discussed above, the last is because the tangent bundle of $\mathbb{R}^q$ is trivial. Now $(\mathfrak{t}_M \times \mathfrak{e}^n_{\mathbb{R}^q})|_{M\times\{0\}} \cong \mathfrak{t}_M \oplus \mathfrak{e}^q_M$ since we have the commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\Delta_M \times 0} & M \times M \times \mathbb{R}^q & \xrightarrow{pr_1} & M \\ {\scriptstyle \mathrm{Id}\times 0}\downarrow & & {\scriptstyle \mathrm{Id}\times\mathrm{Id}\times 0\times\mathrm{Id}}\downarrow & & {\scriptstyle \mathrm{Id}\times 0}\downarrow \\ M \times \mathbb{R}^q & \xrightarrow[\Delta_M\times\Delta_{\mathbb{R}^d}]{} & M \times M \times \mathbb{R}^q \times \mathbb{R}^q & \xrightarrow[pr_{1,3}]{} & M \times \mathbb{R}^q \end{array}$$

Which is a map of microbundles and when we restrict the bottom row to $\{0\} \subseteq \mathbb{R}^q$ we get that the middle map is a homeomorphism. The top row is the sum since the two bundles are just given by the product $(\Delta \times 0)$. Thus we have that $\mathfrak{t}_M$ direct some some trivial bundle is isomorphic to the underlying bundle of $\tau_{M\times\mathbb{R}^q}|_{M\times\{0\}}$, in other words they are stably isomorphic.

($\Leftarrow$) The point is to show that there is a smooth manifold in which $M \times \mathbb{R}^n$ is open, since opens inside smooth manifolds are smooth by restriction. Given an inclusion and a retraction $M \to N \xrightarrow{r} M$ we call this the normal micro-bundle $\mathfrak{n}_{M\to N}$ (if it is locally trivial). Now a trivial normal micro-bundle implies that there is an embedding $M \times \mathbb{R}^k \hookrightarrow N$, just by definition of micro-bundle isomorphism (there is probably some argument needed that the trivial micro-bundle can be taken as the whole V_1). Now assume that for $M \xrightarrow{\xi} BO(q')$ we get that $\mathfrak{t}_M \oplus \mathfrak{e}^q_M \cong \mathcal{U}(\xi)$. Topological manifolds are 'Euclidean neighbourhood retracts' which means that they embed into $\mathbb{R}^n$ as subspaces that are retracts of open sets. So embed $M \subseteq V$ where $V \subseteq \mathbb{R}^k$ is open. Using the retraction $r : V \to M$ we can pullback ξ to a bundle over V. Since V is smooth we can assume that this is itself a smooth vector bundle. This smooth vector bundle by construction restricted to M is stably isomorphic to the tangent micro-bundle. Using that V is open in $\mathbb{R}^k$ you can trivialise the normal bundle of M in the total space of the bundle over V, to get that $M \times \mathbb{R}^{q+k} \subseteq V \times \mathbb{R}^q$ is *open*. Hence it inherits the smooth structure. □

The game here is that an embedding of an open set is only open when the dimensions match.

1.2 Classifying spaces

Recall $\mathrm{Top}(n) = \mathrm{Homeo}_0(\mathbb{R}^n)$ be the topological group of homemorphisms of $\mathbb{R}^n$ that fix the origin, topologised with the compact open topology. There is also $\mathrm{Diff}_0(\mathbb{R}^n)$ a topological group diffeomorphisms that fix the origin, it is classical that, under the Whitney topology:

Lemma ([Kup] Thm 9.1.1s)**.** $O(n) \simeq \mathrm{Diff}_0(\mathbb{R}^n)$

Proof. (Sketch) There is a weak equivalence $\mathrm{O}(m) \hookrightarrow \mathrm{GL}(m)$ by 'Gram-Schmidt' and a weak equivalences $\mathrm{GL}(m) \hookrightarrow \mathrm{Diff}_0(\mathbb{R}^m)$ by 'zooming in on the origin' and 'taking the derivative'. Now to actually prove the result you would need to show that performing Gram-Schmidt is continuous in the matrix and likewise there is more to check in the second case but thats the idea. □

The difficulty is with PL, according to the discussion in [Man16], while the other two groups admit a unique Polish (seperable and completely metrizable) structure, there is no such structure on the group of PL automorphisms of $\mathbb{R}^n$. Moreover it appears to be in contention historically what the 'right topology' should be. The usual definition that seems now to have been adopted is given in terms

of simplical objects and can be made uniformly, which is something of an argument in of itself IT DOESNT APPEAR THAT ANY WORK HAS BEEN DONE ON COMPARING THESE DIFFERENT TOPOLOGIES IN THE PL CASE?.

Definition ([Kup] Def 6.4.1, [ACE$^+$] Ex 21.10)**.** *Let* $\mathrm{CAT}(n)_\bullet$ *be the simplicial group (under composition) whos k simplicies are given by* CAT *isomorphisms* $\Delta^k \times \mathbb{R}^n \to \Delta^k \times \mathbb{R}^n$ *over* Δ^k*. The geometric realisation of a simplicial group is a topological group and this is what we mean by* $\mathrm{CAT}(n) = |\,\mathrm{CAT}(n)_\bullet|$.

These simplicial models are *weakly equivalent* to the Top and Diff models above, not homeomorphic. Moreover to make sense of the smoothness of the maps $\Delta^k \times \mathbb{R}^n$ one would need to say something about manifolds with corners.

On the level of simplicial groups there are clear inclusions $\mathrm{CAT}(n) \hookrightarrow \mathrm{CAT}(n{+}1)$ given by extending by the identity. These induce inclusions $B\,\mathrm{CAT}(n) \hookrightarrow B\,\mathrm{CAT}(n+1)$. Taking the homotopy colimit over these maps defines the stable spaces $\mathrm{Top}, \mathrm{PL}, \mathrm{O}$ and $B\,\mathrm{Top}, B\,\mathrm{PL}, B\mathrm{O}$. Stabilisation is somewhat connected:

Theorem (Precise references needed)**.** *Each of the stabilisation maps are roughly n connected.*

These classifying spaces satisfy the correct theorem.

Theorem ([ACE$^+$] Thm 21.12)**.** *For a* CAT *manifold X there exists a* CAT *micro-bundle* $E\,\mathrm{CAT}(n) \to B\,\mathrm{CAT}(n)$ *such that*

$$\mathrm{Hom}_{\mathrm{Top}}(X, B\,\mathrm{CAT}(n)) \to \mathrm{Bun}^n_{\mathrm{CAT}}(X)$$
$$f \mapsto f^*\big(E\,\mathrm{CAT}(n)\big)$$

induces an isomorphism on π_0*. The right hand side is* CAT *microbundles over X.*

Proof. The classical proof of Milnor shows that for a group G BG classifies bundles with structure group G. We have already seen that CAT $\mathbb{R}^n$ fiber bundles are the same as CAT micro-bundles. □

Again on the level of simplicial sets there are inclusions $\mathrm{O}(n) \hookrightarrow \mathrm{PL}(n) \hookrightarrow \mathrm{Top}(n)$, which induce inclusions on the classifying spaces, on the stable spaces and on the clsasifying spaces of the stable spaces. We use these maps to define the relative spaces. Notice here that there is not a litteral inclusion of diffeomorphisms into PL automorphisms in the same way that there is for orthogonal matricies. Let $\mathrm{CAT}(\bullet)$ be either $\mathrm{CAT}(n)$ or the stable $\mathrm{CAT}(\infty)$.

$$\begin{aligned}
\mathrm{Top}(\bullet)/\,\mathrm{PL}(\bullet) &:= \mathrm{hofib}\Big(B\,\mathrm{PL}(\bullet) \to B\,\mathrm{Top}(\bullet)\Big)\\
\mathrm{Top}(\bullet)/\mathrm{O}(\bullet) &:= \mathrm{hofib}\Big(B\mathrm{O}(\bullet) \to B\,\mathrm{Top}(\bullet)\Big)\\
\mathrm{PL}(\bullet)/\mathrm{O}(\bullet) &:= \mathrm{hofib}\Big(B\mathrm{O}(\bullet) \to B\,\mathrm{PL}(\bullet)\Big)
\end{aligned}$$

In terms of computing the homotopy type, or groups, of the automorphism groups then these fibers are the obstructions in the LES of homotopy groups to the compared groups being isomorphic. Many things are known bout these groups, as summarised by [KS16, V.5]

(1) $TOP_m/PL_m \cong K(Z_2,3)$ for $m \geqslant 5$. See 5.5 .

(2) TOP_m/PL_m is contractible for $m \leqslant 3$. See 5.8 , 5.9 .

(3) $\pi_{k+1+m}(TOP_m/DIFF_m) = \pi_k Aut_{DIFF}(B^m \text{ rel } \partial B^m)$, $m \neq 4$. See 3.4.

$\pi_{k+1+m}(TOP_m/DIFF_m, TOP_{m-1}/DIFF_{m-1}) = \pi_k Aut_{DIFF}(B^{m-1} \times I \text{ re}$
$m \neq 4 \neq m-1$. See 4.6 .

(4) $\pi_k(TOP/DIFF, TOP_m/DIFF_m) = 0$, $k \leqslant m+2$, $m \geqslant 5$,
where $TOP/DIFF = \bigcup_m TOP_m/DIFF_m$. See 4.6 , 5.2 .

(5) $\pi_m(TOP/DIFF) = \Theta_m$, $m \geqslant 5$,
$= \pi_m K(Z_2,3)$, $m \leqslant 6$. See 5.5 .

(6) $\pi_k(TOP_3/DIFF_3) = 0$, $k \leqslant 4$. See 5.8 .

(7) $TOP_m/DIFF_m$ is contractible $m \leqslant 2$. See 5.9 .

Figure 1: Computations of relative groups.

Equation (1) describes the difference between the PL and topological categories and one can see at odd primes these spaces are basically the same and at the even prime they are basically the same in high degrees. Equation (3) is the 'Morlet theorem'. Equation (5) is very useful in my research. In terms of the algebra it is very often useful that:

Theorem ([ACE$^+$] Thm 21.13, [BVP] Thm C)**.** *All the stable spaces described are infinite loop spaces and all the maps described (forgetting) are infinite loop maps. In particular it makes sense to take B of them.*

Proof. there is a functor $\mathrm{CAT}(m) : \mathrm{LI} \to \mathrm{Top}$ sending $\mathbb{R}^m \mapsto \mathrm{CAT}(m)$ and an isometric embedding $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ to the map $g \in \mathrm{CAT}(m) \to \mathrm{CAT}(m)$ given by conjugation

$$g \mapsto \varphi g \varphi^{-1}$$

The conjugation is for functoriality to ensure the identity maps to the identity. $\mathrm{CAT}(\infty)$ is the colimit over these images and hence the functor extends to $\mathbb{R}^\infty$. This defines a map $\mathrm{LI}(\oplus_n \mathbb{R}^\infty, \mathbb{R}^\infty) \to \mathrm{Hom}_{\mathrm{Top}}(\mathrm{CAT}(\infty)^{\otimes n}, \mathrm{CAT}(\infty))$, which is a model for the level wise maps $E_\infty \to \mathrm{End}_{\mathrm{CAT}(\infty)}$. By Mays recognition principle it is therefore an infinite loop space. □

So recall that for fibrations $F \to E \to B$ the Puppe sequence allows us to extend by fibrations to the left, for a cofibration we can extend to the right. For infinite loop spaces and infinite loop maps however we can extend *fibrations* to the *right* by delooping. Thus we can extend the fibration $B\,\mathrm{CAT} \to B\,\mathrm{CAT}'$ to a pair of fibrations

$$B\,\mathrm{CAT} \to B\,\mathrm{CAT}' \xrightarrow{G} B(\mathrm{CAT}' \,/\, \mathrm{CAT})$$

This is the structure that allows us to state whether the (stable) tangent micro-bundle has a smooth lift.

Theorem ([ACE$^+$] Thm 21.15)**.** *Let M be a fixed topological manifold of dimension ≥ 5 with a given* CAT *structure on ∂M. The following are equivalent:*

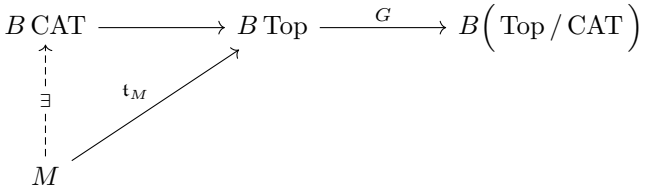
1. *The stable micro-tangent bundle $\mathfrak{t}_M : M \to B\,\mathrm{Top}$ is stably isomorphic to the underlying microbundle of a* CAT *bundle.*
2. *There is a lift $\theta : M \to B\,\mathrm{CAT}$, i.e. $\mathfrak{t}_M \simeq \mathrm{Forget} \circ \theta$.*
3. *The map $G \circ \mathfrak{t}_M : M \to B(\mathrm{Top} \,/\, \mathrm{CAT})$ is null-homotopic.*

Moreover if a lift exists there is a bijection between concordance classes of lifts and $[(M, \partial M), (\mathrm{Top} \,/\, \mathrm{CAT}, *)]$.

Proof. (1) exactly means that there is a map $\xi : M \to B\,\mathrm{CAT}$ such that $\mathcal{U}(\xi) \overset{S}{\cong} \mathfrak{t}_M$.

(2) $\iff$ (1). Let $\xi = \theta$, and note that composing with the forgetful map is 'the same as' taking the underlying microbundle. Basically definitionally.

(3) $\iff$ (2). The righthand map is a fibration



The lefthand map from M is null-homotopic iff it factors over the fiber at the basepoint, namely $B\,\mathrm{CAT}$, by the universal property of a homotopy fiber. □

The stable tangent micro-bundle having a smooth or PL lift is clearly necessary for there to be a smooth or PL structure on the manifold (since the smooth or PL tangent bundle would be such a lift). The hard part now is a sufficient condition.

1.3 Some Categories

Fix a $d \geq 0$, then we can define a topologically enriched category (an ∞-category) of d-dimensional CAT manifolds *without boundary* (as objects) and morphisms being given by $\mathrm{Emb}(M, N)$, the space of CAT embeddings of M into N given the relevant topology topology. We denote this $\mathrm{Man}_d^{\mathrm{CAT}}$. This sits inside the larger category of manifolds with or without boundary that we may occasionally use, the convention there is that $\mathrm{Emb}(M, N)$ sends ∂M to the interior of N.

Let $\mathcal{S}$ denote the ∞-category of spaces. Then $\mathcal{S}/X$ is the ∞-category of spaces over X, that is spaces $Y \to X$ and maps over X.

2 The Main Theorem of Smoothing Theory

2.1 Product Structure Theorem

The first results due to Hirsch-Mazur are the so called 'product strucutre theorem'. If Σ, Σ' are two CAT structures on a topological manifold M then a concordance between them is a CAT structure on $M \times I$ that restricts to Σ, Σ' on each end. An isotopy between Σ, Σ' is a path of homeomorphisms $h_t : M \to M$ that starts at the identity on M and ends at a CAT isomorphism $M_\Sigma \to M_{\Sigma'}$. In particular if there is an isotopy of CAT structures then those strucutres are isomorphic. The first major theorem is that concordance implies isotopy (CAT Cerf theorem). The second is the so called 'product structure theorem':

Theorem ([HM] §4, [ACE$^+$] Thm 22.8)**.** *For a closed topological manifold M of dimension ≥ 5 with a* CAT *structure Θ on $M \times \mathbb{R}^q$ then there is a concordance from Θ to the product structure on $M \times \mathbb{R}^q$ for the standard* CAT *structure on $\mathbb{R}^q$ and some* CAT *structure on M (in particular one exists).*

Proof. (Barely a sketch) Assume that the statement is proven for $q = 1$, for any manifold (not just compact ones). Assume that there is a CAT structure on $M \times \mathbb{R}^n$, we would like a concordance to the product structure. Apply the $q = 1$ case to the manifold with CAT structure $(M \times \mathbb{R}^{n-1}) \times R$ then by the theorem there is a CAT structure on $M \times \mathbb{R}^{n-1}$ and by induction we can then apply a concordance from this CAT structure to the product cat structure. Thus we need only show it for the $q = 1$ case.

So we need to prove a non-compact version for the $q = 1$ case. So assume that $M \times \mathbb{R}$ has a CAT structure. We need to prove a slightly stronger statement: a relative theorem, so fix an open $U \subseteq M$ that has a fixed CAT structure ρ that agrees with the CAT structure on $M \times \mathbb{R}$ (is restricted to). The claim is that we can extend ρ to a CAT strucutre on M and make the CAT structure on $M \times \mathbb{R}$ concordant to the extension of ρ times $\mathbb{R}$. This relative theorem makes sense to prove chart by chart in the following way. Assume the $M = \mathbb{R}^m$ case is proven, then start with U given and consider a chart V_0. Then $V_0 \cong \mathbb{R}^m$ and by the base case we can put a CAT structure on it relative to $U \cap V_0$ and then obtain a concordance to the product structure. Now choose another chart V_1, we apply the same theorem now considering the open set to be $V_1 \cap (U \cup V_0)$ where $(U \cup V_0)$ *is now given the product strucutre*. Now chart by chart we can put product structures on and each time enforce that they agree with the previously done product structures. Since there will be a countable cover by charts this is an induction. Note that the relative part is very important since a priori we know that for the $\mathbb{R}^m$ case there is obviously a CAT structure.

What is left then is to prove the relative theorem for $M = \mathbb{R}^n$ and $q = 1$. This proof is locally writing maps and explicit homotopies. THE WRITE UP IN [ACE+] IS NOT AMAZING, BUT I SHOULD TRY AGAIN ANONTHER TIME.

Applying the 9.19 result we see that it is sufficient for $\mathbf{t}_M$ to have a smooth lift for M to admit a CAT structure. This is only the first theorem in a long line of smoothing theory results and in particular only a sufficient condition. Is it necessary, even if we know a smooth structure exists how many are there, etc. For a necessary condition it seems likely that working stably wont be enough.

2.2 The Main Theorem

Definition. $\mathrm{CAT}_S(M)$ *denotes the spaces of* CAT *structures on* M*, not the automorphisms. It is defined as the simplicial set with simplicies* CAT *structures on* $\Delta^k \times M$ *such that the projection to* Δ^k *is a* CAT *submersion (locally a projection* $p : p(U) \times \mathbb{R}^k \to p(U)$*) .*

Recall that the π_0 of a simplicial set is given by the zero simplicies modulo one simplicies, modulo (the relation generated by) two zero simplicies being the faces of a one simplex. The set of zero simplicies is clearly the set of CAT structures on M. The one simplicies are CAT structures on $M \times I$ with a submersion to I, this is the notion of 'sliced' concordance, every sliced concordance is a concordance, between the two faces. Thus $\pi_0 \mathrm{CAT}_S(M)$ is precisely CAT structures up to isotopy (since concordance implies isotopy).

Definition. *Let* $f : M \to B\,\mathrm{Top}(m)$ *be fixed, then* $\mathrm{Lift}\Big(f \text{ to } B\,\mathrm{CAT}(m)\Big)$ *is defined as a simplicial set with simplicies given by continuous maps* $\sigma : \Delta^k \times M \to B\,\mathrm{CAT}(m)$ *that factor* f*, that is such that* $f\,\mathrm{pr}_M = (\mathrm{forget}) \circ \sigma$*. They dont define the relative lifts, but one can infer that it is compatable lifts of both maps simultaneously.*

So here we see that the set of zero simplicies is the set of lifts or factoriastions of f through $B\,\mathrm{CAT}(m)$. The one simplicies are continuous maps $M \times I \to B\,\mathrm{CAT}(m)$, with submersions to I, that is exactly homotopies *through lifts* between the two ends. Thus $\pi_0 \mathrm{Lift}\Big(f \text{ to } B\,\mathrm{CAT}(m)\Big)$ is precisely lifts of f up to homotopy 'over $B\,\mathrm{Top}(m)$'. Finally we can state what I will call the main theorem of smoothing theory is:

Theorem ([KS16] Essay V. Classification Thm 2.3, Eq 4.4)**.** *Let* M^m *be a topological manifold, such that* $m, m-1 \neq 4$ *(neither* M *nor its boundary is dimension* 4*). There is a homotopy equivalence*

$$\mathrm{CAT}_S(M^n) \simeq \mathrm{Lift}\Big((f, \partial f) \text{ to } (B\,\mathrm{CAT}(m), B\,\mathrm{CAT}(m-1))\Big)$$

Where $f : M \to B\,\mathrm{Top}(m)$ *classifies the tangent microbundle.*

On π_0, because the stabilisation maps are highly connected, the stable classifying spaces are sufficient for the theorem. Moreover since lifts are in bijection with nullhomotopic maps to the relative spaces we get

Theorem ([ACE$^+$] Thm 22.17)**.** *Under the hypotheses, there is a bijection*

$$\pi_0 \operatorname{CAT}_S(M) \cong [(M, \partial M), (\operatorname{Top} / \operatorname{CAT}, *)].$$

I have sketched else where the arduous proof of KS of the 'main theorem' of smoothing theory. What we want to do here is sketch how to translate that statement into the statement of pullbacks.

Theorem. *The following diagram is a pullback*

$$\begin{array}{ccccc}
\operatorname{Man}_d^o & \xrightarrow{forget} & \operatorname{Man}_d^{\mathrm{PL}} & \xrightarrow{forget} & \operatorname{Man}_d^t \\
\downarrow{\scriptstyle T^O(-)} & \lrcorner & \downarrow{\scriptstyle T^{\mathrm{PL}}(-)} & \lrcorner & \downarrow{\scriptstyle T^t(-)} \\
\mathbb{S}/BO(d) & \xrightarrow[forget]{} & \mathbb{S}/B\mathrm{PL}(d) & \xrightarrow[forget]{} & \mathbb{S}/B\mathrm{Top}(d)
\end{array}$$

Proof. So the literal meaning of this statement is that

$$\operatorname{Man}_d^o \simeq \mathbb{S}/BO(d) \underset{\mathbb{S}/B\,\mathrm{Top}(d)}{\times} \operatorname{Man}_d^t$$

Infinity categorically the RHS is triples, $\tau \in \mathbb{S}/BO(d)$, $M \in \operatorname{Man}_d^t$ and a path that is an equivalence $\operatorname{forget}(\tau) \xleftrightarrow{\sim} \tau_M$ to the tangent microbundle. It is clear that

$$\operatorname{Lift}\Big((f, \partial f) \text{ to } (B\operatorname{CAT}(m), B\operatorname{CAT}(m-1))\Big) \simeq \{M\} \underset{\mathbb{S}/B\,\mathrm{Top}(d)}{\times} \mathbb{S}/BO(d),$$

when f classifies the micro-tangent bundle. Hence by smoothing theory

$$\operatorname{Diff}_S(M) \simeq \{M\} \underset{\mathbb{S}/B\,\mathrm{Top}(d)}{\times} \mathbb{S}/BO(d).$$

Thus the fibers of the two categories agree. Now you have to say something about gluing and mapping spaces but honestly I think this is enough for me to say its believable for now.

2.3 Isotopy VS. Diffeomorphism

So first notice that if two smooth structures are isotopic then in particular they are diffeomorphic, since an isotopy is a homotopy (through homeomorphisms) from a diffeomorphism to the identity. On the other hand smooth structures *with an isotopy to the identity* is a more complicated structure than just diffeomorphism classes, since a single smooth structure pair might have multiple diffeomorphisms and multiple isotopies to the identity. So we go from structures with a map (isotopies) to just structures (diffeomorphism), thus we should look to mod out by something.

The pullback

$$\operatorname{Top}(M) \xrightarrow{\varphi^*\Sigma} \operatorname{Diff}_S(M)$$

is well defined since pulling back an atlas by a homeomorphism still gives a smooth atlas because on overlaps the two copies will cancel each other out. On π_0 this induces a group action $\pi_0 \operatorname{Top}(M) \curvearrowright \pi_0\mathrm{Diff}(M)$, and we claim that there is a bijection of sets

$$\operatorname{Diff}_S(M)_0/\mathrm{diffeo} \cong \pi_0\mathrm{Diff}_S(M)/\pi_0 \operatorname{Top}(M).$$

Two structures are going to be diffeomorphic iff there is a homeomorphism such that pulling back a structure gives the other. Then the orbits of the action clearly collect all the isotopy classes of smooth structures together that are connected by pulling back and hence diffeomorphic.

3 Example Applications

3.1 Unique Structures

Theorem ([ACE$^+$] Thm 22.18)**.** *Smooth structures on $\mathbb{R}^n$ for $n \geq 5$ are unique.*

Proof.

$$\pi_0 \operatorname{CAT}_S(\mathbb{R}^n) \cong [\mathbb{R}^n, \operatorname{Top}/O] \cong \pi_0 \operatorname{Top}/\mathrm{O} = 0$$

Where the last fact follows from connectivity of stabilistion and equation (7) of the table. □

As the reference notes this is circular since the product structure theorem uses uniqueness of smooth structures, however this is kind of neat.

3.2 Smoothings on the Sphere and Homotopy Groups of PL/O

The reference for this is again Hirch-Mazur. Let Γ_k is the group of concordance classes of smoothings on the k-sphere.

Theorem ([Hir63] Thm 5.1, [HM] II.4.2, [LR65] Thm 6.3)**.** *There is an isomorphism $\pi_k(\mathrm{PL}/\mathrm{O}) \to \Gamma_k$.*

Proof. The fibration sequence

$$\mathrm{PL}/O \to \mathrm{Top}/O \to \mathrm{Top}/\mathrm{PL}$$

along with equation (1), that $\mathrm{Top}/\mathrm{PL} \simeq K(\mathbb{Z}_2, 3)$ implies using the LES for homotopy groups that for $k \geq 5$ the π_k of these two spaces agree. Hence using the main theorem of smoothing

$$\pi_0 \mathrm{Diff}_S(S^n) \cong [S^n, \operatorname{Top}/\mathrm{O}] \cong \pi_n \operatorname{Top}\mathrm{O} \cong \pi_n \mathrm{PL}/\mathrm{O}.$$

□

Note that $\Gamma_k \cong \Theta_k$ by h-cobordism type arguments. This makes sense since Θ_n is basically *orented* smooth structures on the sphere. Here we have concordance classes, if we go to diffeomorphism then $\pi_0 \mathrm{Diff}_S(S^n)/\mathbb{Z}_2 \cong \mathrm{Diff}_S(S^n)_0/\mathrm{diffeo}$ since $\pi_0 \operatorname{Top}(S^n) \simeq \mathbb{Z}_2$ given by orientation.

3.3 Splitting the Homotopy Sequence

Hirsch-Mazur is the references for these results. It seems that they do not state it explicitly, as Brumfiel does in [Bru68].

Theorem ([LR65] §6)**.** *The homotopy exact sequence of the fibration*

$$\mathrm{PL}/\mathrm{O} \to B\mathrm{O} \to B\,\mathrm{PL}$$

breaks up into SES's

$$0 \to \pi_{k+1}(B\mathrm{O}) \to \pi_{k+1}(B\,\mathrm{PL}) \to \pi_k(\mathrm{PL}/\mathrm{O}) \to 0$$

Proof. Using equation (5) the LES looks like for $k \geq 5$

$$\cdots \to \Theta_k \xrightarrow{f} \pi_k B\mathrm{O} \to \pi_k B\,\mathrm{PL} \to \cdots$$

and we want to see that $f = 0$. We can see that $f(\Sigma) = [\tau_\Sigma] - [\tau_{S^k}] \in \pi_k B\mathrm{O}$, by Kervaire-Milnor every homotopy sphere is *stably* parallelisable and hence $[\tau_\Sigma] = 0$ (both classes are zero). □

3.4 Morlet

The reference is [ST26, Thm 7.3, §9.2]. Morlet follows from the diagram below by applying Alexanders trick and that pullback squares have homotopy equivalent fibers.

Theorem. *The following is a pullback,*

$$\begin{array}{ccc} \mathrm{Diff}_\partial(D^k) & \xrightarrow{D} & \Omega^k \mathrm{O}_k \\ \downarrow & \lrcorner & \downarrow \\ * \simeq \mathrm{PL}_\partial(D^k) & \xrightarrow[D^{\mathrm{PL}}]{} & \Omega^k \mathrm{PL}_k \end{array}$$

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